Exercises

- Analyze and draw the graphs of

$$f(x) = x\sqrt{1-x^2} \quad \text{with} \quad x \in [-1,1]$$

$$f(x) = \sin(x^2 - 4) \quad \text{with} \quad x \in [0,\pi]$$

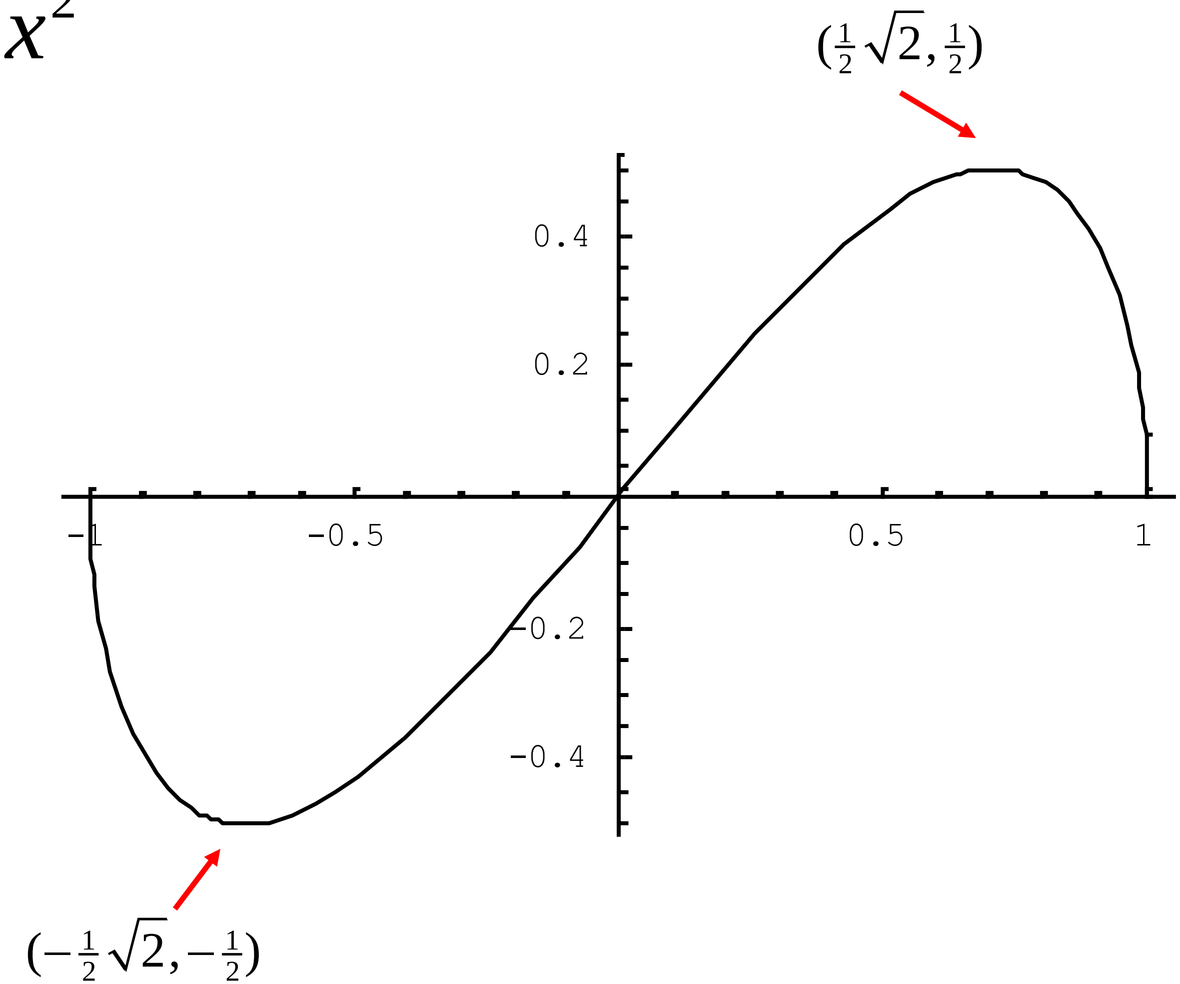
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t^2 \\ t^3 - 3t \end{pmatrix} \quad \text{with} \quad \mathrm{t} \in [-2,2]$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin^3 t \end{pmatrix} \quad \text{with} \quad \mathrm{t} \in [-\pi,\pi]$$

Exercise 1

$$f(x) = x\sqrt{1-x^2}$$

$$x \in [-1,1]$$



Exercise 1

- Find zero crossings

$$x\sqrt{1-x^2}=0 \Leftrightarrow x=0 \vee \sqrt{1-x^2}=0$$
$$\Leftrightarrow x=0 \vee 1-x^2=0$$
$$\Leftrightarrow x=0 \vee x^2=1$$
$$\Leftrightarrow x=0 \vee x=-1 \vee x=1$$

- Look at behavior at domain limits (see zero crossings)
- No singularity

Exercise 1

- Find zero crossings of the derivative and sign around them

$$\left(x\sqrt{1-x^2}\right)' = (x)'\left(\sqrt{1-x^2}\right) + x\left(\sqrt{1-x^2}\right)'$$

$$= \sqrt{1-x^2} + x \times \frac{1}{2\sqrt{1-x^2}} \times (1-x^2)'$$

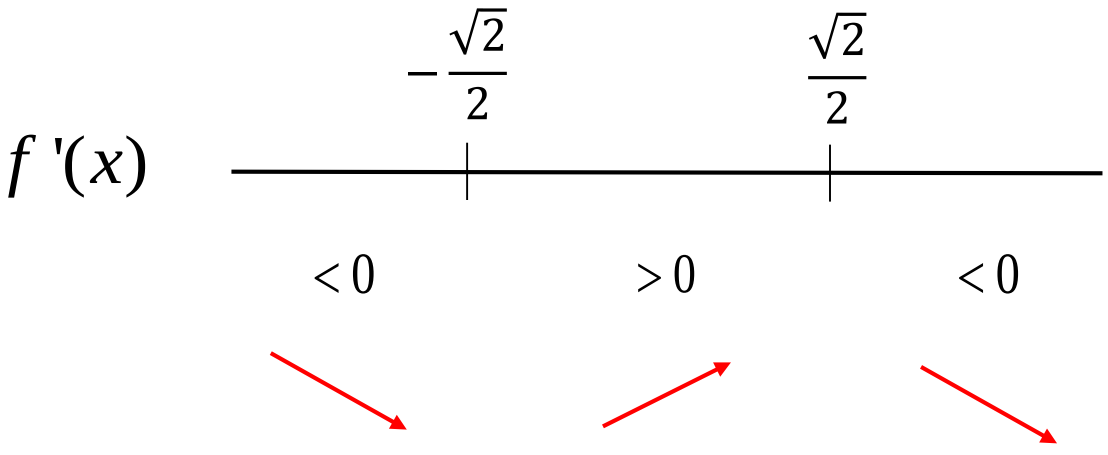
$$= \sqrt{1-x^2} + \frac{x}{2\sqrt{1-x^2}} \times (-2x)$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

equals zero when $x = \pm\frac{\sqrt{2}}{2}$

Exercise 1

- Sign

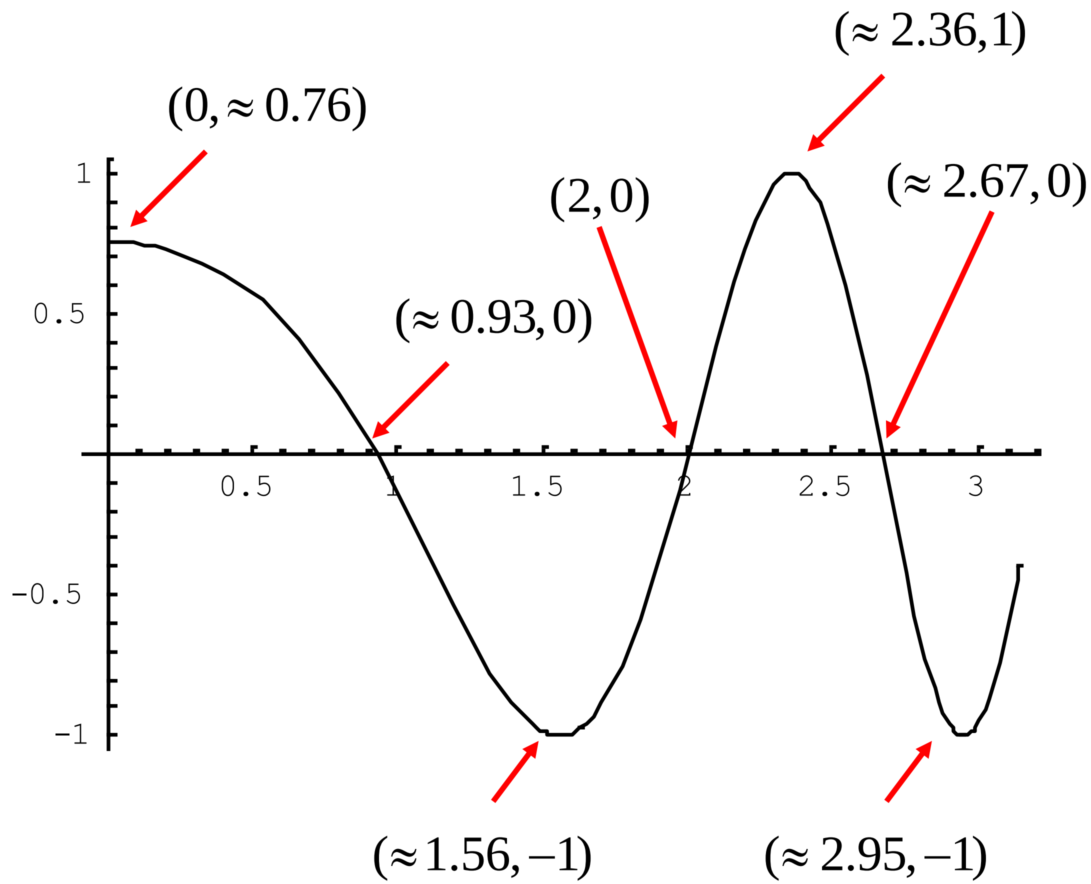


- Minimum at $(-\frac{\sqrt{2}}{2}, -\frac{1}{2})$
- Maximum at $(\frac{\sqrt{2}}{2}, \frac{1}{2})$

Exercise 2

$$f(x) = \sin(x^2 - 4)$$

$$x \in [0, \pi]$$



Exercise 2

- Find zero crossings

$$\sin(x^2 - 4) = 0 \Leftrightarrow x^2 - 4 = 0 + 2k\pi, \text{ for integer } k$$
$$\vee\ x^2 - 4 = \pi - 0 + 2k\pi$$
$$\Leftrightarrow x = \sqrt{4 + 2k\pi} \vee x = \sqrt{4 + \pi + 2k\pi}$$

  - in $[0, \pi]$: $x = 2 \vee x = \sqrt{4 - \pi} \vee x = \sqrt{4 + \pi}$

- Look at behavior at domain limits
  - $f(0) = \sin(-4) \approx 0.757$
  - $f(\pi) = \sin(\pi^2 - 4) \approx -0.402$
- No singularity

Exercise 2

- Find zero crossings of the derivative and sign around them

$(\sin(x^2-4))' = \cos(x^2-4) \times 2x = 2x\cos(x^2-4)$

equals zero if $x = 0 \vee \cos(x^2-4) = 0$

$$x = 0 \vee x^2 = 4 + \frac{\pi}{2} + 2k\pi \vee x^2 = 4 - \frac{\pi}{2} + 2k\pi$$

$$x = 0 \vee x = -\sqrt{\frac{8+\pi}{2} + 2k\pi} \vee x = \sqrt{\frac{8+\pi}{2} + 2k\pi}$$

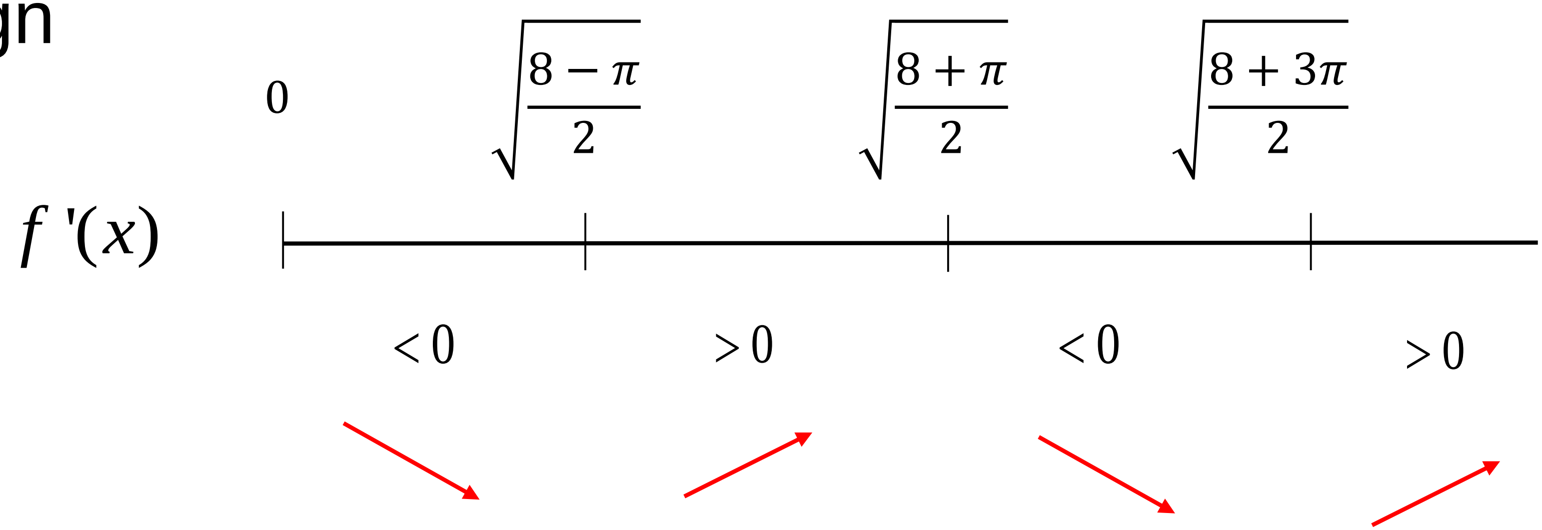
$$\vee\, x = -\sqrt{\frac{8-\pi}{2} + 2k\pi} \vee x = \sqrt{\frac{8-\pi}{2} + 2k\pi}$$

- in $[o, \pi]$:

$$x = 0 \vee x = \sqrt{\frac{8-\pi}{2}} \vee x = \sqrt{\frac{8+\pi}{2}} \vee x = \sqrt{\frac{8+3\pi}{2}}$$

Exercise 2

- Sign

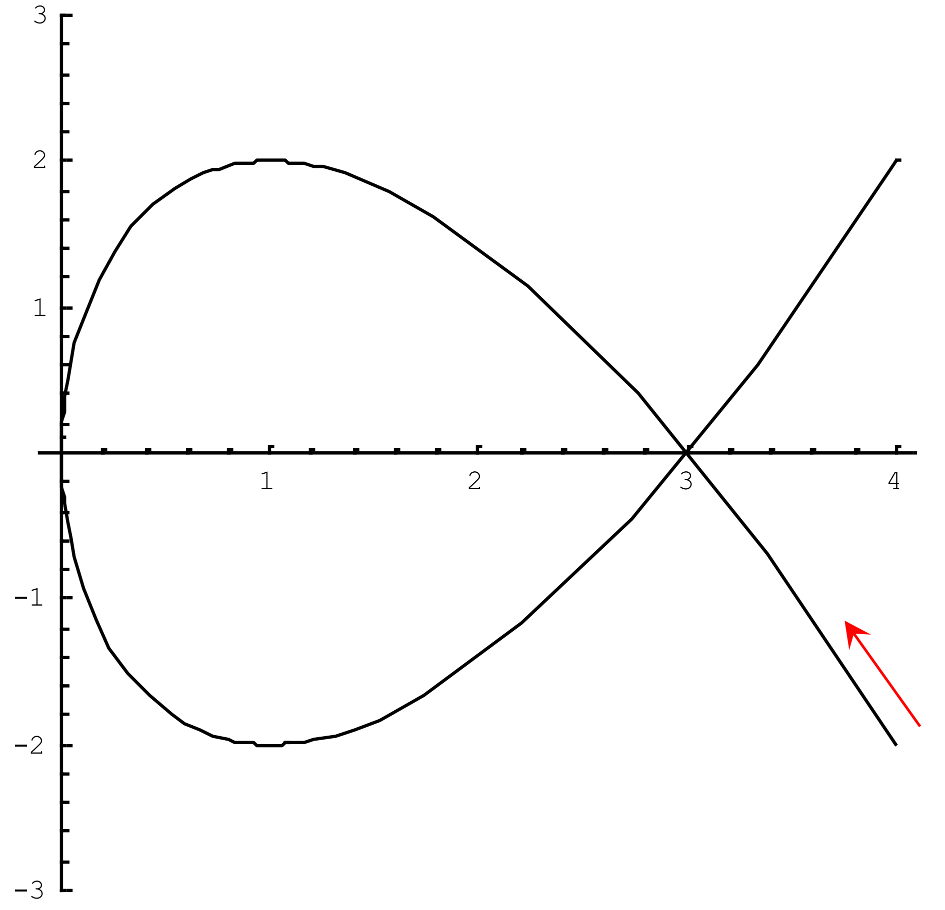


- Local minimum at $\left(\sqrt{\frac{8-\pi}{2}}, -1\right)$ and $\left(\sqrt{\frac{8+3\pi}{2}}, -1\right)$
- Local maximum at $\left(\sqrt{\frac{8+\pi}{2}}, 1\right)$ and $(0, \sin(-4))$

Exercise 3

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t^2 \\ t^3 - 3t \end{pmatrix}$$

$$t \in [-2, 2]$$



Exercise 3

- Find zero crossings of component functions

$$\begin{cases} x(t) = 0 \Leftrightarrow t^2 = 0 \Leftrightarrow t = 0 \\ y(t) = 0 \Leftrightarrow t^3 - 3t = 0 \Leftrightarrow t = 0 \lor t^2 = 3 \Leftrightarrow t = 0 \lor t = \pm\sqrt{3} \end{cases}$$

  - *f* crosses x-axis at $t = 0$
  - *f* crosses y-axis at $t = 0, t = -\sqrt{3}$ and $t = \sqrt{3}$
  - so *f* pass by origin at $t = 0$

- Look at behavior at domain ends
  - $\begin{pmatrix} x(-2)= 4 \\ y(-2)=-2 \end{pmatrix}$; $\begin{pmatrix} x(2)=4 \\ y(2)=2 \end{pmatrix}$

- No singularity

Exercise 3

- Look at derivative

$$\begin{pmatrix} x'(t) = (t^2)' = 2t \\ y'(t) = (t^3 - 3t)' = 3t^2 - 3 \end{pmatrix}$$

  - zero crossing

$$\begin{pmatrix} 2t = 0 \Leftrightarrow t = 0 \\ 3t^2 - 3 = 0 \Leftrightarrow t = -1 \vee t = 1 \end{pmatrix}$$

  - tangent vector at 0, -1 and 1

$$\begin{pmatrix} x'(0) = 0 \\ y'(0) = -3 \end{pmatrix}$$

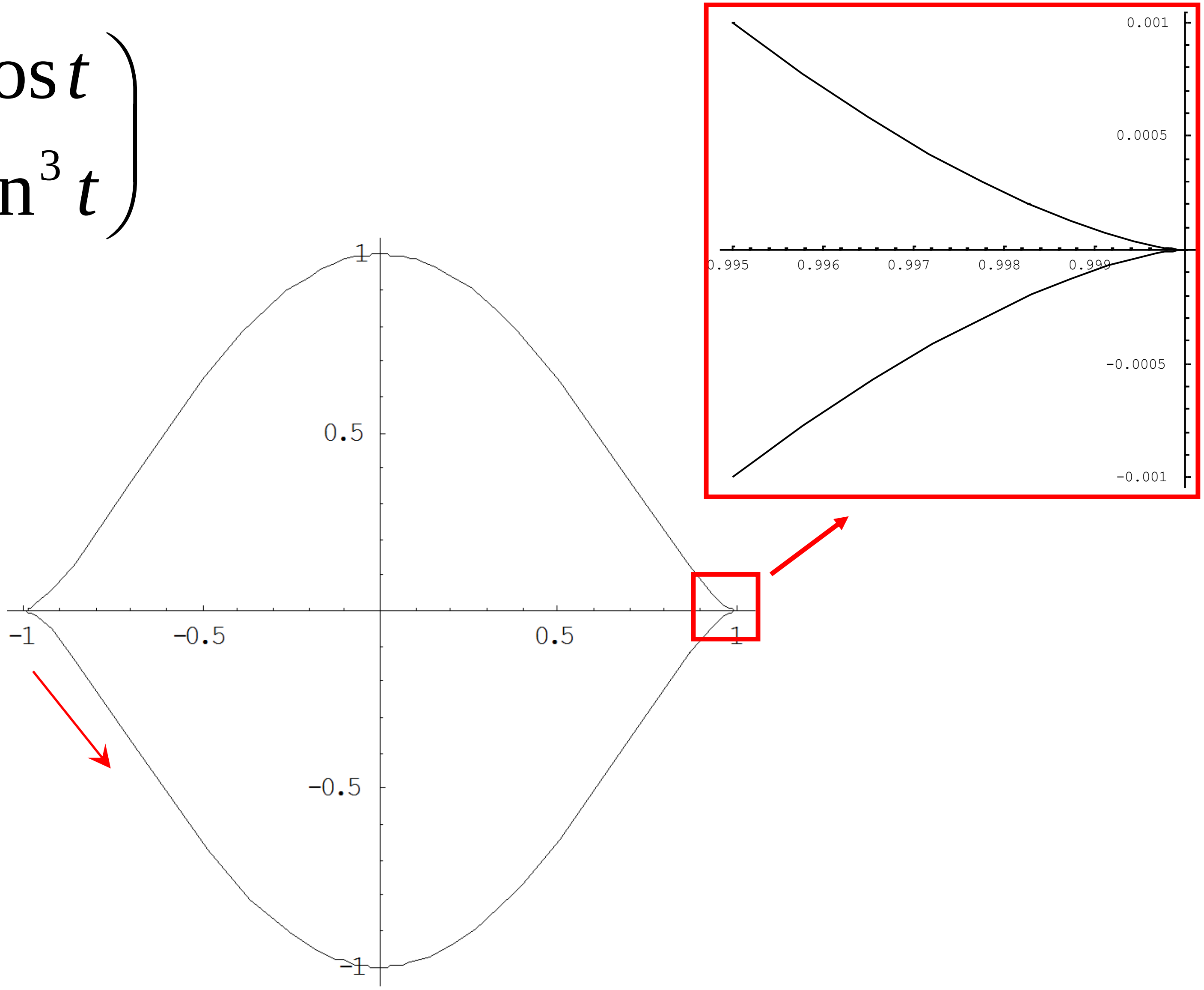
$$\begin{pmatrix} x'(-1) = -2 \\ y'(-1) = 0 \end{pmatrix}$$

$$\begin{pmatrix} x'(1) = 2 \\ y'(1) = 0 \end{pmatrix}$$

Exercise 4

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin^3 t \end{pmatrix}$$

$$t \in [-\pi, \pi]$$



Exercise 4

- Find zero crossings of component functions

$$\begin{cases} x(t) = 0 \Leftrightarrow \cos t = 0 \Leftrightarrow t = \frac{\pi}{2} + 2k\pi \vee t = -\frac{\pi}{2} + 2k\pi, \text{for integer k} \\ y(t) = 0 \Leftrightarrow \sin^3(t) = 0 \Leftrightarrow t = 2k\pi \vee t = \pi + 2k\pi,\ \text{for integer k} \end{cases}$$

  - *f* crosses x-axis at $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$
  - *f* crosses y-axis at $t = -\pi$, $t = 0$ and $t = \pi$
  - so *f* does not pass by origin
- Look at behavior at domain ends
  - $\begin{pmatrix} x(-\pi)=-1 \\ y(-\pi)=0 \end{pmatrix}$; $\begin{pmatrix} x(\pi)=-1 \\ y(\pi)=0 \end{pmatrix}$

Exercise 4

- Look at derivative

$$\begin{pmatrix} x'(t) = (\cos t)' = -\sin t \\ y'(t) = (sin^3(t))' = 3\, sin^2(t) \cos t \end{pmatrix}$$

  - zero crossing

$$\begin{pmatrix} -\sin t = 0 \Leftrightarrow t = 2k\pi \vee t = \pi + 2k\pi, \text{for integer k} \\ 3\, sin^2(t) \cos t = 0 \Leftrightarrow sin^2(t) = 0 \vee \cos t = 0 \end{pmatrix}$$

$$\begin{pmatrix} t = 2k\pi \vee t = \pi + 2k\pi, \text{for integer k} \\ t = 2k\pi \vee t = \pi + 2k\pi \vee t = \frac{\pi}{2} + 2k\pi \vee t = -\frac{\pi}{2} + 2k\pi \end{pmatrix}$$

  - tangent vector at $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

$$\begin{pmatrix} x'(-\pi) = 0 \\ y'(-\pi) = 0 \end{pmatrix}; \begin{pmatrix} x'\left(-\frac{\pi}{2}\right) = 1 \\ y'\left(-\frac{\pi}{2}\right) = 0 \end{pmatrix}; \begin{pmatrix} x'(0) = 0 \\ y'(0) = 0 \end{pmatrix}; \begin{pmatrix} x'\left(\frac{\pi}{2}\right) = -1 \\ y'\left(\frac{\pi}{2}\right) = 0 \end{pmatrix};$$

$$\begin{pmatrix} x'(\pi) = 0 \\ y'(\pi) = 0 \end{pmatrix}$$